RAMAKRISHNA MISSION VIDYAMANDIRA

(A Residential Autonomous College under University of Calcutta)

P.G. First Year, Second Semester (January – June), 2011
Mid-Semester Examination, March, 2011

MATHEMATICS

Date : 21 March 2011 — Full Marks : 100
Time : 11am – 3pm

[Use separate Answer Script for each teacher]

Group A (Paper V)

Unit I : SKC

Answer **any three** questions : [5×3 = 15]

1. Prove that a finite extension of a field is algebraic. Cite example to justify that an algebraic extension need not be finite.
2. Show that the number $5-3\sqrt{2}$ is algebraic over the field Q of all rational numbers. Hence establish that $Q(5-3\sqrt{2}) = Q(\sqrt{2})$.
3. Prove that an element α is algebraic over a field F if and only if $[F(\alpha):F]$ is finite.
4. Let f(x) = $x^4 - 2x^2 - 2 \in Q[x]$. Find the roots α, β of f(x) such that $Q(\alpha) \cong Q(B)$. Determine the splitting field of f(x).
5. Let α be a root of $x^p - x - 1$ over a field F of characteristics p. Show that $F(\alpha)$ is a separable extension of F.
6. Prove that for any given prime p and any given positive integer n there always exists a finite field with p^n number of elements. Construct a field with 27 elements.

Unit II : PSM

7. Answer **any two** questions : [2×5 = 10]
   a) State and prove the 'universal property' of a free R-module F(A) on a set A.
   b) Define torsion-free R-module. Prove that the R-module R is torsion-free if and only if R is an integral domain. Show that, a finite abelian group G is not a Q-module if |G| > 1.
   c) Prove that Klein 4-group is a vector space over $\frac{Z}{2Z}$. Let M be an R-module. If S is a non-empty subset of M, define the annihilation of S in R and show that $Ann_R(S)$ is a left ideal of R. Show further that whenever S is a submodule of M, the annihilator $Ann_R(S)$ becomes a two sided ideal.

Group B (Paper VI) : SB

8. Define a ring. Show that the null set belongs to every ring. Prove that any ring of sets is closed with respect to finite intersection and symmetric difference. Convers by, also show that any non-void class of sets which is closed with respect to finite intersection and also symmetric difference is a ring. [2+1+2+3 = 8]

OR,

Define a semi ring. Prove that the collection $P = \{[a,b): a,b \in R\}$ is a semi-ring of subsets of R. Is this collection a ring? Prove that in a semi-ring S, it $A \in S$ and also $A_1, A_2, ..., A_n \in S$, then $A \setminus \bigcup_{i=1}^{n} A_i$ can be written as a finite union of disjoint sets of S. [2+2+1+3 = 8]

9. Define a monotone class. Is a σ-ring a monotone class? Prove that if a monotone class contains a ring, then it also contains the generated σ-ring. [2+3+4 = 9]

OR,

Define a Dynkin system. Is a σ-ring a Dynkin system? Prove that if a Dynkin system contains a class of sets closed with respect to finite intersection, then it also contains the σ-ring generated by that class. [2+3+4 = 9]

10. Define measure on a ring. Prove that if ε is a nonempty class of sets and μ is a measure on the ring $R(\varepsilon)$ generated by ε such that $\mu(E) < \infty$ whenever $E \in \varepsilon$, then μ is finite on $R(\varepsilon)$. If μ is an extended real valued, non negative, and additive set function define on a ring R and such that μ(E) for at least one set E in R, then $\mu(\phi) = 0$. [2+3+3 = 8]

OR,

State what is meant when we say that an extended real valued set function on a class of sets is subtractive. Prove that a measure on a ring is subtractive. Prove that for a measure μ on a ring R, if $\{E_n\}$ is a increasing sequence of sets in R for which $\lim\limits_{n\to\infty} E_n \in R$, then $\mu(\lim\limits_{n\to\infty} E_n) = \lim\limits_{n\to\infty} \mu(E_n)$. [2+3+3 = 8]

Group C (Paper VII) : PKC

11. Answer **any two** questions : [2×2 = 4]

a) Prove that Wronskian, of two linearly dependent differentiable functions vanishes identically.

b) Prove that no nontrivial solution of the differential equation $\ddot{x} + q(t)x = 0, (q(t) \le 0)$, can have more than one zero.

c) Prove that all eigen values of : $\frac{d}{dt}[p(t)\dot{u}] + q(t)u + \lambda u = 0$

where $u(a)\cos\alpha - p(a)\dot{u}(a)\sin\alpha = 0,\ u(b)\cos\beta - p(b)\dot{u}(b)\sin\beta = 0,$ are real.

d) Prove that the solutions of $\dot{x} = A(t)x$ form a vector space V over R, where $\begin{bmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ . & . & . & . \\ . & . & . & . \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{bmatrix}$.

12. Answer **any two** questions : [2×6 = 12]

a) Let f satisfy a Lipschitz condition on D and let $\varphi_1(t)$ and $\varphi_2(t)$ be two solutions of the differential equation $\dot{x} = f(t,x)$ on the domain D. Suppose both $\varphi_1(t)$ and $\varphi_2(t)$ are defined on the open interval I. If $\varphi_1(\sigma) = \varphi_2(\sigma)$ for some $\sigma \in I$, then prove that $\varphi_1(t) = \varphi_2(t)$ for every t in I.

b) State and prove Gronwall Inequality.

c) Let u(t) and v(t) be nontrivial solutions of the differential equations $\ddot{x} + p(t)x = 0$ and $\ddot{x} + q(t)x = 0$, respectively where $p(t) \ge q(t)$. Prove that u(t) vanish at least once between any two consecutive zeros of v(t), unless p(t) = q(t) and u is a constant multiple of u.

13. Answer **any one** question : [1×9 = 9]

a) State and prove Ascoli-Arzela Theorem.

b) If f(t,x) is continuous on the open set $D \subset R \times R^n$, prove that for each point $(\sigma, \xi) \in D$, there exists at least one solution to the initial-value problem $\dot{x} = f(t,x)$, $\xi = x(\sigma)$.

GROUP D (PAPER VIII) : KPD

Answer **Question No. 1** and **any three** from the rest :

14. Answer **any two** of the following : [2×2 = 4]

    a) What are functions and functionals?

    b) How the convergence of a sequence of test functions $\{\varphi_n\}$ to a function φ belonging to D is defined?

    c) How a generalized function is defined?

    d) Which functional defines the generalized function $P\frac{1}{x}$?

15. Show that $\frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \to \delta(x)$ as $L \to 0+$ in $D'(R^1)$. [7]

16. What are regular and singular generalized functions? Show that Dirac's delta function is a singular generalized function. [7]

17. Explain how the product of a generalized function $f \in D'$ by an infinitely differential function a(x) is defined? Show that $a(x)\delta(x) = a(0)\delta(x)$. [7]

18. a) Find the Fourier transform of $\frac{1}{x}$ assuming that this transform exists.

    b) Show $\theta'(x) = \delta(x)$, where $\theta(x)$ is the unit step function. [4+3]

19. If the function f(x) be such that in any finite interval it is continuous and its derivative is piecewise continuous and the integrals $\int_{-\infty}^{\infty} |f(x)|$ and $\int_{-\infty}^{\infty} |x f(x)| dx$ are convergent, then show that $F[f'(x)] = -ikF(k)$, where F(k) is the Fourier transform of f(x). [7]

◆◆❖◆◆